

The Harnack inequality and related properties for solutions to elliptic and parabolic equations with divergence-free lower-order coefficients

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To the memory of M.S. Birman

1 Introduction

Qualitative properties of solutions to partial differential equations are intensively studied over last half of century. In this paper we deal with classical properties, namely, strong maximum principle, Hölder estimates, the Harnack inequality and the Liouville Theorem.

We consider elliptic and parabolic equations of divergence type:

$$\mathcal{L}u \equiv -D_i(a_{ij}(x)D_ju) + b_i(x)D_iu = 0; \quad (\mathbf{DE})$$

$$\mathcal{M}u \equiv \partial_t u - D_i(a_{ij}(x; t)D_ju) + b_i(x; t)D_iu = 0. \quad (\mathbf{DP})$$

We mostly deal with *a priori* estimates for Lipschitz generalized (sub/super)solutions. When these estimates are established, we discuss the possibility of their generalization for weak (sub/super)solutions. In this case we assume $Du \in L_{2,loc}(\Omega)$ in **(DE)** and $u \in L_{2,\infty,loc}(Q)$, $Du \in L_{2,loc}(Q)$ in **(DP)**.

We always suppose that operators under consideration are uniformly elliptic (parabolic), i.e. for all values of arguments

$$\nu|\xi|^2 \leq a_{ij}(\cdot)\xi_i\xi_j \leq \nu^{-1}|\xi|^2, \quad \xi \in \mathbb{R}^n, \quad (1)$$

where ν is a positive constant.

The properties of generalized solutions to the equations **(DE)**–**(DP)** were investigated in a number of papers. Hölder estimates for solutions of **(DE)** were obtained by E. De Giorgi [DG] for $\mathbf{b} \equiv 0$ and by C. Morrey [M] for \mathbf{b} belonging to the Morrey space lying between L_n and any L_q , $q > n$ (\mathbf{b} stands for (b_i)). Corresponding result for **(DP)** was established by J. Nash [Na] for $\mathbf{b} \equiv 0$ and by O.A. Ladyzhenskaya and N.N. Ural'tseva [LU1] for $\mathbf{b} \in L_{q+2}$, $q > n$.

Harnack's inequality for operators without lower-order coefficients was proved by J. Moser ([Mo1] for **(DE)** and [Mo2] for **(DP)**). N. Trudinger [Tru] proved it for **(DE)** with $\mathbf{b} \in L_q$, $q > n$. G. Lieberman (see [Li1, Ch. VI]) extended the result of [Mo2] for $\mathbf{b} \in L_{q,\ell}$, $\frac{n}{q} + \frac{2}{\ell} < 1$. Obviously, Harnack's inequality implies Hölder estimates. Also strong maximum principle follows from Harnack's inequality and weak maximum principle. Some sharpening of mentioned results, as well as corresponding results for nondivergence equations, are discussed in our preprint [NU].

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In this paper we consider mainly the equations **(DE)** and **(DP)** with additional structure condition

$$\operatorname{div}(\mathbf{b}) \leq 0 \quad \text{in the sense of distributions.} \quad (2)$$

The equations with the lower-order coefficients satisfying this structure condition arise in some applications (see, e.g., [Z], [KNSS], [CSTY], [SSSZ]). We show that in this case the assumptions on \mathbf{b} can be considerably weakened in the scale of Morrey spaces.

Our paper is organized as follows. In Section 2 we deal with elliptic equations. Section 3 is devoted to parabolic equations (recall that only two-sided Liouville's Theorem holds for these equations). In Section 4 we show an application of our results to some equations arising in hydrodynamics. We underline that this Section contains just exemplary instances, and we make no pretence to the novelty of results. In particular, the statements of Theorems 4.1 and 4.3 are in fact obtained in [KNSS].

Let us recall some notation. $x = (x_1, \dots, x_n)$ is a vector in \mathbb{R}^n , $n \geq 2$, with the Euclidean norm $|x|$; $(x; t)$ is a point in \mathbb{R}^{n+1} .

Ω is a domain in \mathbb{R}^n and $\partial\Omega$ is its boundary. For a cylinder $Q = \Omega \times]0, T[$ we denote by $\partial''Q = \partial\Omega \times]0, T[$ its lateral surface and by $\partial'Q = \partial''Q \cup \{\overline{\Omega} \times \{0\}\}$ its parabolic boundary.

We define

$$\begin{aligned} B_R(x^0) &= \{x \in \mathbb{R}^n : |x - x^0| < R\}, & B_R &= B_R(0); \\ Q_R^{\lambda, \theta}(x^0; t^0) &= B_{\lambda R}(x^0) \times]t^0 - \theta R^2; t^0[, & Q_R^{\lambda, \theta} &= Q_R^{\lambda, \theta}(0; 0), \quad Q_R = Q_R^{1, 1} \end{aligned}$$

(note that $Q_{\lambda R} = Q_R^{\lambda, \lambda^2}$).

The indices i, j vary from 1 to n . Repeated indices indicate summation.

The symbol D_i denotes the operator of differentiation with respect to x_i ; in particular, $Du = (D_1u, \dots, D_nu)$ is the gradient of u . $\partial_t u$ stands for the derivative of u with respect to t .

The dashed integral stands for the mean value: $\int_E u = (\operatorname{meas} E)^{-1} \int_E u$.

We denote by $\|\cdot\|_{p, \Omega}$ the norm in $L_p(\Omega)$. We introduce a scale of anisotropic spaces $L_{q, \ell}(Q) = L_\ell([0, T] \rightarrow L_q(\Omega))$ with the norm $\|f\|_{q, \ell, Q} = \| \|f(\cdot; t)\|_{q, \Omega} \|_{\ell, [0, T]}$. Obviously, $L_{q, q}(Q) = L_q(Q)$.

We also introduce a scale of Morrey spaces

$$\mathbb{M}_q^\alpha(\Omega) = \{f \in L_q(\Omega) : \|f\|_{\mathbb{M}_q^\alpha(\Omega)} \equiv \sup_{B_\rho(x) \subset \Omega} \rho^{-\alpha} \|f\|_{q, B_\rho(x)} < \infty\}.$$

Parabolic Morrey spaces $\mathbb{M}_{q, \ell}^\alpha(Q)$ are introduced in a similar way, using $Q_\rho(x; t) \subset Q$ instead of $B_\rho(x) \subset \Omega$.

Finally, we introduce the space $\mathcal{V}(Q)$ of weak solutions to **(DP)** with the norm defined by

$$\|f\|_{\mathcal{V}(Q)}^2 = \|f\|_{2, \infty, Q}^2 + \|Df\|_{2, 2, Q}^2.$$

We set $f_+ = \max\{f, 0\}$, $f_- = \max\{-f, 0\}$, $\operatorname{osc}_\Omega f = \sup_\Omega f - \inf_\Omega f$. For $1 \leq p < n$, $p^* = \frac{np}{n-p}$ is the Sobolev conjugate to p .

We use letters N, C (with or without indices) to denote various constants. To indicate that, say, N depends on some parameters, we list them in the parentheses: $N(\dots)$.

2 Elliptic case

Recall that u is a (Lipschitz) subsolution of the equation $\mathcal{L}u = 0$ in Ω (here \mathcal{L} is an operator of the form **(DE)**), if for any Lipschitz test function $\eta \geq 0$, supported in Ω ,

$$\int_\Omega (a_{ij} D_j u D_i \eta + b_i D_i u \eta) dx \leq 0.$$

We take $\eta = \varphi'(u) \cdot \xi$, where ξ is a nonnegative Lipschitz function, supported in $B_{\lambda R} \subset \Omega$, while $\varphi \in \mathcal{C}^2(\mathbb{R})$ is a convex function vanishing in \mathbb{R}_- . This gives

$$\int_{B_{\lambda R} \cap \{u > 0\}} \left(a_{ij} D_j v D_i \xi + \frac{\varphi''(u)}{\varphi'^2(u)} a_{ij} D_j v D_i v \xi + b_i D_i v \xi \right) dx \leq 0, \quad (3)$$

$v = \varphi(u)$.

Then, by mollification at a neighborhood of the origin, one can weaken in (3) the assumption $\varphi \in \mathcal{C}^2(\mathbb{R})$ to $\varphi \in \mathcal{C}^2(\mathbb{R}_+ \cup \mathbb{R}_-)$.

Lemma 2.1. *Let \mathcal{L} be an operator of the form (DE) in $B_{\lambda R}(x^0)$, $\lambda > 1$, and let the conditions (1) and (2) be satisfied. Let also $\mathbf{b} \in L_q(B_{\lambda R}(x^0))$ with some $\frac{n}{2} < q \leq n^1$.*

Then there exists a positive constant N_1 depending on n, ν, λ, q and the quantity

$$\mathcal{N} = \mathcal{N}(R, \lambda) \equiv R^{1-\frac{n}{q}} \|\mathbf{b}\|_{q, B_{\lambda R}(x^0)},$$

such that any Lipschitz subsolution of the equation $\mathcal{L}u = 0$ in $B_{\lambda R}(x^0)$ satisfies

$$\sup_{B_R(x^0)} u_+ \leq N_1 \left(\int_{B_{\lambda R}(x^0)} u_+^2 dx \right)^{\frac{1}{2}}. \quad (4)$$

Proof. We use classical technique of Moser (see, e.g., [LU2, Ch.IX]). Without loss of generality, we assume $x^0 = 0$.

We put in (3) $\varphi(\tau) = \tau_+^p$, $p > 1$, and $\xi = v\zeta^2$ where ζ is a smooth cut-off function in $B_{\lambda R}$. Then we obtain

$$\int_{B_{\lambda R}} \left(\frac{2p-1}{p} a_{ij} D_j v D_i v \zeta^2 + 2a_{ij} D_j v v D_i \zeta \zeta + b_i D_i v v \zeta^2 \right) dx \leq 0. \quad (5)$$

The last term in (5) can be estimated using (2) and the Hölder inequality:

$$- \int_{B_{\lambda R}} b_i D_i v v \zeta^2 dx \leq \int_{B_{\lambda R}} b_i v^2 \zeta D_i \zeta dx \leq \|\mathbf{b}\|_{q, B_{\lambda R}} \|v\zeta\|_{r, B_{\lambda R}}^{2-\frac{1}{s}} \|v\zeta^{1-s} |D\zeta|^s\|_{2, B_{\lambda R}}^{\frac{1}{s}}, \quad (6)$$

where $s > 2$ is defined by $\frac{1}{s} = 1 - \frac{n}{2q}$ while $r = \frac{2(2q+n)}{2q+n-4}$. Note that $2 < r < 2^*$, and, by the embedding theorem,

$$\|v\zeta\|_{r, B_{\lambda R}} \leq C(n)(\lambda R)^{n(\frac{1}{r}-\frac{1}{2^*})} \|D(v\zeta)\|_{2, B_{\lambda R}} \leq C(n)(\lambda R)^{\frac{1}{2s-1}} \left(\|Dv\zeta\|_{2, B_{\lambda R}} + \|vD\zeta\|_{2, B_{\lambda R}} \right). \quad (7)$$

Using (1), (6) and (7), we obtain from (5)

$$\begin{aligned} \|Dv\zeta\|_{2, B_{\lambda R}}^2 &\leq \frac{1}{\nu} \int_{B_{\lambda R}} a_{ij} D_j v D_i v \zeta^2 dx \leq C_1(n, \nu, s, \lambda) \times \\ &\times \left[\|Dv\zeta\|_{2, B_{\lambda R}} \|vD\zeta\|_{2, B_{\lambda R}} + R^{\frac{1}{s}} \|\mathbf{b}\|_{q, B_{\lambda R}} \left(\|Dv\zeta\|_{2, B_{\lambda R}}^{\frac{2-1}{s}} + \|vD\zeta\|_{2, B_{\lambda R}}^{\frac{2-1}{s}} \right) \|v\zeta^{1-s} |D\zeta|^s\|_{2, B_{\lambda R}}^{\frac{1}{s}} \right], \end{aligned}$$

¹For $q = n$, the assumption (2) can be removed. We discuss this at the end of this Section.

and therefore

$$\|Dv\zeta\|_{2,B_{\lambda R}} \leq C_2(n, \nu, s, \lambda) \cdot \left[\|vD\zeta\|_{2,B_{\lambda R}} + R\|\mathbf{b}\|_{q,B_{\lambda R}}^s \|v\zeta^{1-s}|D\zeta|^s\|_{2,B_{\lambda R}} \right]. \quad (8)$$

We put $R_m = R(1 + 2^{-m}(\lambda - 1))$, $m \in \mathbb{N} \cup \{0\}$, and substitute $\zeta = \zeta_m$ such that

$$\zeta_m \equiv 1 \text{ in } B_{R_{m+1}}; \quad \zeta_m \equiv 0 \text{ out of } B_{R_m}; \quad \frac{|D\zeta_m|}{\zeta_m^{1-\frac{1}{s}}} \leq \frac{2^m C_3(s)}{(\lambda - 1)R}.$$

Then (8) implies

$$\|Dv\zeta_m\|_{2,B_{R_m}} \leq \frac{C_4(n, \nu, s, \lambda)}{R} \cdot \|v\|_{2,B_{R_m}} \cdot (2^m + (2^m \mathcal{N})^s). \quad (9)$$

Now for $p = p_m \equiv (\frac{r}{2})^m$ we obtain from (7) and (9)

$$\begin{aligned} \left(\int_{B_{R_{m+1}}} u_+^{2p_{m+1}} dx \right)^{\frac{1}{2p_{m+1}}} &\leq \left(C(n) \int_{B_{R_m}} (v\zeta_m)^r dx \right)^{\frac{1}{rp_m}} \leq \\ &\leq \left(2^{2ms} C_5 \int_{B_{R_m}} v^2 dx \right)^{\frac{1}{2p_m}} = \left(2^{2ms} C_5 \int_{B_{R_m}} u_+^{2p_m} dx \right)^{\frac{1}{2p_m}}, \end{aligned} \quad (10)$$

where C_5 depends only on n, ν, λ, s and \mathcal{N} .

Iterating (10) we arrive at (4). \square

Corollary 2.1. *Let \mathcal{L} satisfy the assumptions of Lemma 2.1 in $B_{\lambda R}(x^0)$. If a Lipschitz subsolution of $\mathcal{L}u = 0$ in $B_{\lambda R}(x^0)$ satisfies*

$$\text{meas}(\{u > k\} \cap B_{\lambda R}(x^0)) \leq \mu \text{meas}(B_{\lambda R}), \quad \mu < N_1^{-2}, \quad (11)$$

for some k , then

$$\sup_{B_R(x^0)} (u - k) \leq N_1 \sqrt{\mu} \sup_{B_{\lambda R}(x^0)} (u - k), \quad (12)$$

(here N_1 is the constant from Lemma 2.1).

Proof. We apply Lemma 2.1 to $u - k$. \square

We need the following variant of the embedding theorem.

Proposition A. *Let $1 \leq p < n$. Suppose that a non-negative function $u \in W_p^1(B_R)$ vanishes on a positive measure set \mathcal{E}_0 . Let $\eta = \eta(|x|)$ be a non-decreasing function, $0 \leq \eta \leq 1$, and $\eta|_{\mathcal{E}_0} \equiv 1$. Then, for any $1 \leq q \leq p^*$ and for any measurable set $\mathcal{E} \subset B_R$,*

$$\|u\eta\|_{q,\mathcal{E}} \leq \frac{C(n)R^n}{\text{meas}(\mathcal{E}_0)} \text{meas}^{\frac{1}{q}-\frac{1}{p^*}}(\mathcal{E}) \cdot \|Du\eta\|_{p,B_R}.$$

Proof. For $q = p = 1$ this Proposition was proved in [LSU, Ch. II, Lemma 5.1]. In this Lemma the following inequality was obtained:

$$\text{meas}(\mathcal{E}_0) \cdot u(x) \eta(x) \leq \frac{(2R)^n}{n} \int_{B_R} \frac{|Du(y)| \eta(y)}{|y - x|^{n-1}} dy.$$

By the Hardy–Littlewood–Sobolev inequality (see, e.g., [LL, Sec. 4.3]), we get

$$\text{meas}(\mathcal{E}_0) \cdot \|u\eta\|_{p^*,B_R} \leq C(n,p)R^n \cdot \|Du\eta\|_{p,B_R},$$

and the statement follows by Hölder inequality. \square

Lemma 2.2. *Let \mathcal{L} satisfy the assumptions of Lemma 2.1 in $B_{\lambda R}(x^0)$. Then for any $\delta > 0$ there exists a positive constant β depending on $n, \nu, \lambda, q, \delta$ and the quantity \mathcal{N} , such that if a Lipschitz nonnegative supersolution of $\mathcal{L}V = 0$ in $B_{\lambda R}(x^0)$ satisfies*

$$\text{meas}(\{V \geq k\} \cap B_R(x^0)) \geq \delta \cdot \text{meas}(B_R) \quad (13)$$

for some $k > 0$, then

$$\inf_{B_R(x^0)} V \geq \beta k. \quad (14)$$

Proof. Without loss of generality, we can assume $V > 0$; otherwise we deal with $V + \varepsilon$ and pass to the limit as $\varepsilon \downarrow 0$. Also we put $x^0 = 0$.

We define $u = 1 - \frac{V}{k}$. Note that $u < 1$ is a subsolution, and therefore, we can apply the relation (3) with φ defined only for $\tau < 1$.

We put in (3) $\varphi(\tau) = \ln_-(1 - \tau)$. This gives for $v = \varphi(u)$

$$\int_{B_{\lambda R}} \left(a_{ij} D_j v D_i \xi + a_{ij} D_j v D_i v \xi + b_i D_i v \xi \right) dx \leq 0. \quad (15)$$

We substitute into (15) $\xi = \zeta^2$ where ζ is a smooth cut-off function that equals 1 in $B_{\frac{1+\lambda}{2}R}$. Then, using (1), (2) and the Hölder inequality, we obtain

$$\begin{aligned} \|Dv \zeta\|_{2, B_{\lambda R}}^2 &\leq \frac{1}{\nu} \int_{B_{\lambda R}} a_{ij} D_j v D_i v \zeta^2 dx \leq \frac{2}{\nu} \int_{B_{\lambda R}} \left(-a_{ij} D_j v \zeta D_i \zeta + b_i v \zeta D_i \zeta \right) dx \leq \\ &\leq \frac{2}{\nu} (\|Dv \zeta\|_{2, B_{\lambda R}} \|D\zeta\|_{2, B_{\lambda R}} + \|\mathbf{b}\|_{q, B_{\lambda R}} \|v \zeta\|_{q', B_{\lambda R}} \|D\zeta\|_{\infty, B_{\lambda R}}). \end{aligned}$$

Note that v vanishes on the set $\{V \geq k\} \cap B_R$. Therefore, we can estimate the last term by Proposition A. By (13), this gives

$$\|Dv \zeta\|_{2, B_{\lambda R}} \leq C_6(n, \nu, \lambda, q, \delta) R^{\frac{n}{2}-1} \cdot (1 + \mathcal{N}).$$

Applying Proposition A once more, we obtain

$$\left(\int_{B_{\frac{1+\lambda}{2}R}} v^2 dx \right)^{\frac{1}{2}} \leq C_7,$$

where C_7 depends only on $n, \nu, \lambda, q, \delta$ and \mathcal{N} .

Finally, the relation (15) implies that v is a subsolution. So, we apply Lemma 2.1 to v in $B_{\frac{1+\lambda}{2}R}$ and arrive at the estimate $\sup_{B_R} v_+ \leq C_8 \equiv N_1 C_7$, which is equivalent to (14) with $\beta = \exp(-C_8)$. \square

Corollary 2.2 (strong maximum principle). *Let \mathcal{L} be an operator of the form (DE) in Ω , and let the conditions (1) and (2) be satisfied. Let also $\mathbf{b} \in L_{q, \text{loc}}(\Omega)$ with some $\frac{n}{2} < q \leq n$. Then any Lipschitz nonconstant supersolution of $\mathcal{L}V = 0$ in Ω cannot attain its minimum at interior point of Ω .*

Proof. Assume the converse. Without loss of generality, $\inf_{\Omega} V = 0$. Then there exists $x^0 \in \Omega$ which is a frontier point of the set $\{V > 0\}$. Choose R such that $\overline{B_{2R}}(x^0) \subset \Omega$. Then the relation (13) holds for some $k > 0$ and $\delta > 0$, and we obtain (14), a contradiction. \square

Lemma 2.3. *Let \mathcal{L} satisfy the assumptions of Lemma 2.1 in B_{3R} . Then any Lipschitz solution of $\mathcal{L}u = 0$ in B_{3R} satisfies the estimate*

$$\operatorname{osc}_{B_R} u \leq \varkappa_0 \operatorname{osc}_{B_{3R}} u, \quad (16)$$

where $\varkappa_0 < 1$ depends on n, ν, q and the quantity \mathcal{N} .

Proof. We set

$$k = \frac{1}{2} \left(\sup_{B_{3R}} u + \inf_{B_{3R}} u \right)$$

and consider two cases.

1. Let the relation (11) hold with $\lambda = 2$ and $\mu = \frac{1}{4}N_1^{-2}$. Then, by Corollary 2.1,

$$\sup_{B_R} u \leq \frac{1}{2} \left(\sup_{B_{2R}} u + k \right) \leq \sup_{B_{3R}} u - \frac{1}{4} \operatorname{osc}_{B_{3R}} u.$$

2. In the opposite case we apply Lemma 2.2 (with $\lambda = \frac{3}{2}$ and $\delta = \mu$) to the (non-negative) function $V = u - \inf_{B_{3R}} u$. This gives $\inf_{B_R} V \geq \inf_{B_{2R}} V \geq \beta(k - \inf_{B_{3R}} u)$, and thus,

$$\inf_{B_R} u \geq \inf_{B_{3R}} u + \frac{\beta}{2} \operatorname{osc}_{B_{3R}} u.$$

In both cases we arrive at (16) with $\varkappa_0 = \min \left\{ \frac{1}{4}, \frac{\beta}{2} \right\}$. \square

Corollary 2.3 (Hölder estimate). *Let \mathcal{L} satisfy the assumptions of Lemma 2.1 in B_{R_0} . Let also $\sup_{R < R_0} \mathcal{N}(R, 1) < \infty$.*

Then any Lipschitz solution of $\mathcal{L}u = 0$ in B_{R_0} satisfies the estimate

$$\operatorname{osc}_{B_\rho} u \leq N_2 \left(\frac{\rho}{r} \right)^\gamma \cdot \operatorname{osc}_{B_r} u, \quad 0 < \rho < r \leq R_0, \quad (17)$$

where N_2 and γ depend on n, ν, q and $\sup_{R < R_0} \mathcal{N}(R, 1)$.

Proof. Iterating the estimate (16) we arrive at (17) with $\gamma = -\log_3(\varkappa_0)$. \square

Corollary 2.4 (two-sided Liouville's theorem). *Let \mathcal{L} be an operator of the form (DE) in \mathbb{R}^n , and let the conditions (1) and (2) be satisfied. Let also $\mathbf{b} \in L_{q, \text{loc}}(\mathbb{R}^n)$, with some $\frac{n}{2} < q \leq n$. Finally, assume that*

$$\liminf_{R \rightarrow \infty} \widehat{\mathcal{N}}(R, 1) < \infty. \quad (18)$$

Then any Lipschitz bounded solution of $\mathcal{L}u = 0$ in \mathbb{R}^n is a constant.

Remark 1. If $\mathbf{b} \in L_q(\mathbb{R}^n)$, then (18) is obviously satisfied.

Proof. Iteration of (16) with respect to a suitable sequence $R_m \rightarrow \infty$ gives the statement. \square

Lemma 2.4. *Let \mathcal{L} be an operator of the form (DE) in B_{2R} , and let the conditions (1) and (2) be satisfied. Let also $\mathbf{b} \in \mathbb{M}_q^{\frac{n}{q}-1}(B_{2R})$ with some $\frac{n}{2} < q \leq n$.*

Let for a Lipschitz nonnegative supersolution of $\mathcal{L}V = 0$ in B_{2R} and for some $y \in B_{2R}$, the inequality $\inf_{B_\rho(y)} V = k > 0$ holds with $\rho = \frac{1}{4} \operatorname{dist}(y, \partial B_{2R})$. Then

$$\inf_{B_R} V \geq \widehat{\beta} \left(\frac{\rho}{R} \right)^{\widehat{\gamma}} k, \quad (19)$$

where $\widehat{\beta}$ and $\widehat{\gamma}$ depend on n, ν, q and $\|\mathbf{b}\|_{\mathbb{M}_q^{\frac{n}{q}-1}(B_{2R})}$.

Proof. Denote by \mathfrak{N} an integer number such that $2^{-(\mathfrak{N}+1)}R < \rho \leq 2^{-\mathfrak{N}}R$ and consider a ball $B_{\mathfrak{r}_0}(y^0)$, where $\mathfrak{r}_0 = 2^{-\mathfrak{N}}R$, $y^0 = 2R(1 - 2^{-\mathfrak{N}})\mathbf{e}$ and $\mathbf{e} = \frac{y}{|y|}$. It is easy to see that $B_{\mathfrak{r}_0}(y^0) \subset B_{3\rho}(y)$, and by Lemma 2.2 (with $\lambda = \frac{4}{3}$ and $\delta = \frac{1}{3^n}$),

$$\inf_{B_{\mathfrak{r}_0}(y^0)} V \geq \inf_{B_{3\rho}(y)} V \geq \beta k. \quad (20)$$

Now we introduce the sequence of balls $B_{\mathfrak{r}_m}(y^m)$, $m = 1, \dots, \mathfrak{N}$, as follows:

$$\mathfrak{r}_m = 2\mathfrak{r}_{m-1}, \quad y^m = y^{m-1} - \mathfrak{r}_m \mathbf{e}.$$

For all $m = 1, \dots, \mathfrak{N}$ one has

$$B_{2\mathfrak{r}_m}(y^m) \subset B_{2R}; \quad \text{meas}(B_{\mathfrak{r}_{m-1}}(y^{m-1}) \cap B_{\mathfrak{r}_m}(y^m)) \geq C(n) \cdot \text{meas}(B_{\mathfrak{r}_m}).$$

Thus, Lemma 2.2 (with $\lambda = 2$ and $\delta = C(n)$) gives

$$\inf_{B_{\mathfrak{r}_m}(y^m)} V \geq \beta \cdot \inf_{B_{\mathfrak{r}_{m-1}}(y^{m-1})} V.$$

Since $B_{\mathfrak{r}_{\mathfrak{N}}}(y^{\mathfrak{N}}) = B_R$, we obtain

$$\inf_{B_R} V \geq \beta^{\mathfrak{N}} \cdot \inf_{B_{\mathfrak{r}_0}(y^0)} V \geq \left(\frac{\rho}{R}\right)^{\widehat{\gamma}} \cdot \inf_{B_{\mathfrak{r}_0}(y^0)} V,$$

where $\widehat{\gamma} = -\log_2(\beta)$.

Combining this estimate with (20), we arrive at (19). \square

Theorem 2.5 (the Harnack inequality). *Let \mathcal{L} satisfy the assumptions of Lemma 2.4 in B_{2R} . Then there exists a positive constant N_3 depending on n , ν , q and $\|\mathbf{b}\|_{\mathbb{M}_q^{\frac{n}{q}-1}(B_{2R})}$, such that any Lipschitz nonnegative solution of $\mathcal{L}u = 0$ in B_{2R} satisfies*

$$\sup_{B_R} u \leq N_3 \cdot \inf_{B_R} u. \quad (21)$$

Proof. We follow the idea of Safonov ([S2]). Denote by $y \in B_{2R}$ a maximum point of the function

$$v(x) = (\text{dist}(x, \partial B_{2R}))^{\widehat{\gamma}} \cdot u(x)$$

(here $\widehat{\gamma}$ is the constant from Lemma 2.4) and set

$$\rho = \frac{1}{4} \text{dist}(y, \partial B_{2R}); \quad \mathfrak{M} = v(y) = (4\rho)^{\widehat{\gamma}} \cdot u(y).$$

It is obvious that

$$\sup_{B_R} u \leq \frac{\mathfrak{M}}{R^{\widehat{\gamma}}} = \left(\frac{4\rho}{R}\right)^{\widehat{\gamma}} \cdot u(y); \quad (22)$$

$$\sup_{B_{2\rho}(y)} u \leq \frac{\mathfrak{M}}{(2\rho)^{\widehat{\gamma}}} = 2^{\widehat{\gamma}} \cdot u(y). \quad (23)$$

Denote $k = \frac{1}{2}u(y)$. If $\text{meas}(\{u > k\} \cap B_{2\rho}(y)) \leq \mu \text{meas}(B_{2\rho})$, then Corollary 2.1 (with $\lambda = 2$) and (23) imply the relation

$$k = u(y) - k \leq \sup_{B_{\rho}(y)} (u - k) \leq N_1 \sqrt{\mu} \sup_{B_{2\rho}(y)} (u - k) \leq N_1 \sqrt{\mu} (2^{\widehat{\gamma}+1} - 1)k,$$

which is impossible for $\mu \leq \mu_0 \equiv \frac{1}{2^{2\gamma+2}} N_1^{-2}$. Thus, $\text{meas}(\{u > k\} \cap B_{2\rho}(y)) \geq \mu_0 \text{meas}(B_{2\rho})$, and Lemma 2.2 (with $\lambda = 2$ and $\delta = \mu_0$) gives

$$\inf_{B_\rho(y)} u \geq \inf_{B_{2\rho}(y)} u \geq \beta k = \frac{\beta}{2} \cdot u(y). \quad (24)$$

Finally, Lemma 2.4 gives

$$\inf_{B_R} u \geq \widehat{\beta} \left(\frac{\rho}{R} \right)^{\widehat{\gamma}} \inf_{B_\rho(y)} u. \quad (25)$$

Combining (22), (24) and (25), we arrive at (21) with $N_3 = \frac{2^{2\gamma+1}}{\beta\widehat{\beta}}$. \square

Theorem 2.6 (one-sided Liouville's theorem). *Let \mathcal{L} be an operator of the form (DE) in \mathbb{R}^n , and let the conditions (1) and (2) be satisfied. Let also $\mathbf{b} \in \mathbb{M}_{q,loc}^{\frac{n}{q}-1}(\mathbb{R}^n)$ with some $\frac{n}{2} < q \leq n$, and for some $\delta > 0$*

$$\liminf_{R \rightarrow \infty} \sup_{|x|=R} \|\mathbf{b}\|_{\mathbb{M}_q^{\frac{n}{q}-1}(B_{\delta R}(x))} < \infty. \quad (26)$$

Then any Lipschitz semibounded solution of $\mathcal{L}u = 0$ in \mathbb{R}^n is a constant.

Remark 2. If $\mathbf{b} \in \mathbb{M}_q^{\frac{n}{q}-1}(\mathbb{R}^n)$, then (26) is obviously satisfied.

Proof. Without loss of generality, we can assume that u is bounded from below, and $\inf_{\mathbb{R}^n} u = 0$.

We take a sequence $R_m \rightarrow \infty$ such that $\mathfrak{B} \equiv \sup_m \sup_{|x|=R_m} \|\mathbf{b}\|_{\mathbb{M}_q^{\frac{n}{q}-1}(B_{\delta R}(x))} < \infty$. Further, we cover the sphere $|x| = 1$ with a finite set of balls $B_{\frac{\delta}{2}}(x)$ and dilate these balls to the covering of the sphere $|x| = R_m$. Applying Theorem 2.5 to all the balls of this covering, we obtain $\sup_{|x|=R_m} u \leq C(n, \nu, q, \mathfrak{B}, \delta) \cdot \inf_{|x|=R_m} u$ for any m . By Corollary 2.2,

$$\sup_{B_R} u = \sup_{|x|=R} u; \quad \inf_{|x|=R} u = \inf_{B_R} u \rightarrow 0 \quad \text{as } R \rightarrow \infty,$$

and the statement follows. \square

Let us discuss briefly the possibility to generalize all previous statements for weak (sub/super)solutions.

The proof of Lemma 2.1 runs without changes² also for weak subsolutions of $\mathcal{L}u = 0$ if the bilinear form

$$\mathcal{B}\langle u, \eta \rangle \equiv \int_{B_{\lambda R}} b_i D_i u \eta \, dx$$

can be continuously extended to the pair $(v, v\zeta^2)$ with $Dv \in L_2(B_{\lambda R})$. This is certainly true provided

$$|\mathcal{B}\langle u, \eta \rangle| \leq C \|Du\|_{2, B_{2\lambda R}} \|D\eta\|_{2, B_{2\lambda R}}, \quad u, \eta \in \mathcal{C}_0^\infty(B_{2\lambda R}).$$

It is shown in [MV] that the last estimate holds if

$$\Delta^{-1} \text{rot}(\mathbf{b}) \in BMO^{n \times n}(B_{2\lambda R}); \quad h = |\nabla(\Delta^{-1} \text{div}(\mathbf{b}))|^2 \in \mathfrak{M}_+^{1,2} \quad (27)$$

²More formally, in this case the inequality (3) holds under additional condition that φ is globally Lipschitz. Thus, to derive (8) one should take $\varphi'(u_+) = p \min\{u_+, N\}^{p-1}$, $\xi = \min\{u_+, N\}^p \zeta^2$, $N > 0$, and then pass to the limit as $N \rightarrow \infty$.

(here we assume \mathbf{b} extended by zero); the last notation means a class of so-called admissible weights, i.e.

$$\int_{B_{2\lambda R}} h|v|^2 dx \leq C \|Dv\|_{2, B_{2\lambda R}}^2, \quad v \in \mathcal{C}_0^\infty(B_{2\lambda R}).$$

If $\mathbf{b} \in \mathbb{M}_q^{\frac{n}{q}-1}(B_{\lambda R})$, the first relation in (27) follows from elliptic coercive estimates and the Poincaré inequality, see, e.g., [Tro]. Thus, Lemma 2.1 and, therefore, all subsequent statements hold true for weak (sub/super)solutions of $\mathcal{L}u = 0$ if, for example, $\mathbf{b} \in \mathbb{M}_q^{\frac{n}{q}-1}$ and $\operatorname{div}(\mathbf{b}) \equiv 0$.

In addition, let us consider the case $q = n$. The space $\mathbb{M}_q^{\frac{n}{q}-1}(\Omega)$ now becomes conventional Lebesgue space $L_n(\Omega)$, and we claim that main results of this section hold true without the assumption (2)³. Note that in this case the relation (5) is fulfilled for any weak subsolution u .

First, let $n \geq 3$. Then we estimate the last term in (5) by the Hölder inequality and the Sobolev inequality and obtain an analog of (8):

$$\|Dv \zeta\|_{2, B_{\lambda R}} \leq C'_2(n, \nu) \cdot \left[\|v D\zeta\|_{2, B_{\lambda R}} + \|\mathbf{b}\|_{n, B_{\lambda R}} \|Dv \zeta\|_{2, B_{\lambda R}} \right].$$

If $\|\mathbf{b}\|_{n, B_{\lambda R}} \leq \varepsilon(n, \nu) \equiv (2C'_2)^{-1}$, then for $p = p_m \equiv 2\left(\frac{n}{n-2}\right)^m$ we obtain an analog of (10):

$$\left(\int_{B_{R_{m+1}}} u_+^{2p_{m+1}} dx \right)^{\frac{1}{2p_{m+1}}} \leq \left(C'_5 \int_{B_{R_m}} u_+^{2p_m} dx \right)^{\frac{1}{2p_m}},$$

where C'_5 depends only on n, ν , and λ . The remainder of the proof of Lemma 2.1 runs without changes.

Similarly we prove Lemmas 2.2 and 2.4 for weak supersolutions of $\mathcal{L}V = 0$ under the same assumption $\|\mathbf{b}\|_{n, B_{\lambda R}} \leq \varepsilon(n, \nu)$ (in Lemma 2.4 $\lambda = 2$).

In the case $n = 2$ we use the Yudovich–Pohozhaev embedding theorem (see, e.g., [BIN, 10.6]) instead of the Sobolev inequality. This gives us Lemmas 2.1, 2.2, 2.4 under the assumption $\|\mathbf{b} \ln^{\frac{1}{2}}(1 + \lambda R|\mathbf{b}|)\|_{2, B_{\lambda R}(x^0)} \leq \varepsilon(n, \nu)$.

Further, strong maximum principle holds without smallness assumptions on \mathbf{b} . Indeed, one can choose R sufficiently small such that these assumptions are fulfilled.

Since the proof of Theorem 2.5 depends only on Lemmas 2.1, 2.2, 2.4, the Harnack inequality evidently holds under smallness assumption on \mathbf{b} . However, we can exclude this assumption using a trick of M.V. Safonov.[S3]

Theorem 2.5' (the Harnack inequality). *Let \mathcal{L} be an operator of the form (DE) in B_{2R} , and let the condition (1) be satisfied. Suppose also that*

$$\|\mathbf{b}\|_{n, B_{2R}} \leq \mathfrak{B} \quad \text{for } n \geq 3; \quad \|\mathbf{b} \ln^{\frac{1}{2}}(1 + R|\mathbf{b}|)\|_{2, B_{2R}} \leq \mathfrak{B} \quad \text{for } n = 2. \quad (28)$$

Then there exists a positive constant N'_3 depending only on n, ν and \mathfrak{B} such that any nonnegative weak solution of $\mathcal{L}u = 0$ in B_{2R} satisfies

$$\sup_{B_R} u \leq N'_3 \cdot \inf_{B_R} u. \quad (29)$$

³Moreover, in this case the assumption $\mathbf{b} \in L_n$ can be weakened in the scale of Lorentz spaces to $\mathbf{b} \in \Lambda_{n, q}$ with any $q < \infty$. We do not discuss it here for the reason of place.

Proof. We split the spherical layer $B_{2R} \setminus B_R$ to M layers of equal thickness $\frac{R}{M}$ and put $\delta = \frac{1}{2M}$. Obviously, one can choose M depending only on n, ν and \mathfrak{B} such that at least for one of these layers (say, $K = \{r - 2\delta R < |x| < r + 2\delta R\}$) the following estimates hold:

$$\|\mathbf{b}\|_{n,K} \leq \varepsilon \quad \text{for } n \geq 3; \quad \|\mathbf{b} \ln^{\frac{1}{2}}(1 + 2\delta R|\mathbf{b}|)\|_{2,K} \leq \varepsilon \quad \text{for } n = 2$$

(here $\varepsilon = \varepsilon(n, \nu)$ is the above smallness constant).

We cover the sphere $|x| = r$ with a finite set of balls $B_{\delta R}(x)$ (note that the number of balls depends only on δ). Since all doubled balls $B_{2\delta R}(x)$ lie in K , we can apply Harnack's inequality in these balls. This gives $\sup_{|x|=r} u \leq C(n, \nu, \delta) \cdot \inf_{|x|=r} u$. However, by the maximum principle,

$$\inf_{B_R} u \geq \inf_{|x|=r} u; \quad \sup_{B_R} u \leq \sup_{|x|=r} u,$$

and the statement follows. \square

The following statement can be proved by verbatim repetition of the proof of Theorem 2.6, using Theorem 2.5'.

Theorem 2.6' (the Liouville theorem). *Let \mathcal{L} be an operator of the form (DE) in \mathbb{R}^n . Let the condition (1) be satisfied, and*

$$\mathbf{b} \in L_{n,loc}(\mathbb{R}^n) \quad \text{for } n \geq 3; \quad \mathbf{b} \ln^{\frac{1}{2}}(1 + |\mathbf{b}|) \in L_{2,loc}(\mathbb{R}^2) \quad \text{for } n = 2.$$

Suppose also that for some $\delta > 0$

$$\liminf_{R \rightarrow \infty} \sup_{|x|=R} \|\mathbf{b}\|_{n,B_{\delta R}(x)} < \infty, \quad n \geq 3;$$

$$\liminf_{R \rightarrow \infty} \sup_{|x|=R} \|\mathbf{b} \ln^{\frac{1}{2}}(1 + R|\mathbf{b}|)\|_{2,B_{\delta R}(x)} < \infty, \quad n = 2. \quad (30)$$

Then any weak semibounded solution of $\mathcal{L}u = 0$ in \mathbb{R}^n is a constant.

Remark 3. If $\mathbf{b} \in L_n(\mathbb{R}^n)$ (respectively, $\mathbf{b} \ln^{\frac{1}{2}}(1 + |x||\mathbf{b}|) \in L_2(\mathbb{R}^2)$), then (30) is obviously satisfied.

As for Hölder estimates for solutions, we have two possibilities. The first one is to take in the proof of Lemma 2.3 R sufficiently small, such that the smallness assumptions on \mathbf{b} are satisfied in B_{2R} . This gives the estimate (17) with γ depending only on n and ν , while N_2 depends also on the moduli of continuity of \mathbf{b} in $L_n(B_{R_0})$ (respectively, of $\mathbf{b} \ln^{\frac{1}{2}}(1 + R|\mathbf{b}|)$ in $L_2(B_{R_0})$). The second possibility is to use Theorem 2.5'. This gives (17) with both γ and N_2 depending on n, ν and $\|\mathbf{b}\|_{n,B_{R_0}}$ (respectively, $\|\mathbf{b} \ln^{\frac{1}{2}}(1 + R_0|\mathbf{b}|)\|_{2,B_{R_0}}$).

3 Parabolic case

Lemma 3.1. *Let \mathcal{M} be an operator of the form (DP) in $Q_R^{\lambda,\theta}(x^0; t^0)$, $\lambda > 1$, $\theta > 0$, and let the conditions (1) and (2) be satisfied. Let also $\mathbf{b} \in L_{q,\ell}(Q_R^{\lambda,\theta}(x^0; t^0))$, with some q and ℓ such that*

$$\alpha = \alpha(q, \ell) \equiv \frac{n}{q} + \frac{2}{\ell} - 1 \in [0, 1[\quad (31)$$

(q as well as ℓ may be infinite⁴).

⁴For $\alpha = 0$, the assumption (2) can be removed, with some limitation in the case $q = n$, $\ell = \infty$. We discuss this at the end of this Section.

Then there exists a positive constant N_4 depending on $n, \nu, \lambda, \theta, q, \ell$ and the quantity

$$\widehat{\mathcal{N}} = \widehat{\mathcal{N}}(R, \lambda, \theta) \equiv R^{-\alpha} \|\mathbf{b}\|_{q, \ell, Q_R^{\lambda, \theta}(x^0; t^0)}$$

such that any Lipschitz subsolution of the equation $\mathcal{M}u = 0$ in $Q_R^{\lambda, \theta}(x^0; t^0)$ satisfies

$$\sup_{Q_R^{1, \frac{\theta}{2}}(x^0; t^0)} u_+ \leq N_4 \left(\int_{Q_R^{\lambda, \theta}(x^0; t^0)} u_+^2 dx dt \right)^{\frac{1}{2}}. \quad (32)$$

Remark 4. The quantity $\widehat{\mathcal{N}}$ depends also on q and ℓ . However, we assume these parameters hold fixed, and we do not indicate this dependence. We also do not indicate the dependence $\widehat{\mathcal{N}}$ on x^0 and t^0 .

Proof. We proceed similarly to Lemma 2.1. Without loss of generality, we assume $(x^0; t^0) = (0; 0)$.

For a nonnegative test function η we have

$$\int_{Q_R^{\lambda, \theta}} (\partial_t u \eta + a_{ij} D_j u D_i \eta + b_i D_i u \eta) dx dt \leq 0.$$

We take $\eta = \varphi'(u) \cdot \xi$, where ξ is a cut-off function, Lipschitz in x and vanishing at the neighborhood of $\partial' Q_R^{\lambda, \theta}$, while $\varphi \in \mathcal{C}^2(\mathbb{R})$ is a convex function vanishing in \mathbb{R}_- . This gives

$$\int_{Q_R^{\lambda, \theta} \cap \{u > 0\}} \left(\partial_t v \xi + a_{ij} D_j v D_i \xi + \frac{\varphi''(u)}{\varphi'^2(u)} a_{ij} D_j v D_i v \xi + b_i D_i v \xi \right) dx dt \leq 0, \quad (33)$$

where $v = \varphi(u)$.

As in Section 2, by mollification at a neighborhood of the origin, one can weaken in (3) the assumption $\varphi \in \mathcal{C}^2(\mathbb{R})$ to $\varphi \in \mathcal{C}^2(\mathbb{R}_+ \cup \mathbb{R}_-)$.

Now we put in (33) $\varphi(\tau) = \tau_p^p$, $p > 1$, and $\xi = \chi_{\{t < \bar{t}\}} \cdot v \zeta^2$ where ζ is a smooth cut-off function in $Q_R^{\lambda, \theta}$, $\bar{t} \in] - \theta R^2, 0[$. Then we obtain

$$\begin{aligned} & \frac{1}{2} \int_{B_{\lambda R}} (v \zeta)^2 \Big|^{t=\bar{t}} dx + \\ & + \int_{Q_R^{\lambda, \theta}} \chi_{\{t < \bar{t}\}} \left(\frac{2p-1}{p} a_{ij} D_j v D_i v \zeta^2 + 2a_{ij} D_j v v D_i \zeta \zeta - v^2 \zeta \partial_t \zeta + b_i D_i v v \zeta^2 \right) dx dt \leq 0. \end{aligned} \quad (34)$$

The last term in (34) can be estimated using (2) and the Hölder inequality:

$$\begin{aligned} - \int_{Q_R^{\lambda, \theta}} \chi_{\{t < \bar{t}\}} b_i D_i v v \zeta^2 dx dt & \leq \int_{Q_R^{\lambda, \theta}} \chi_{\{t < \bar{t}\}} b_i v^2 \zeta D_i \zeta dx dt \leq \\ & \leq \|\mathbf{b}\|_{q, \ell, Q_R^{\lambda, \theta}} \|v \zeta\|_{r, l, Q_R^{\lambda, \theta}}^{2-\frac{1}{s}} \|v \zeta^{1-s} |D \zeta|^s\|_{2, 2, Q_R^{\lambda, \theta}}^{\frac{1}{s}}, \end{aligned} \quad (35)$$

where $s > 2$ is defined by $\frac{1}{s} = 1 - \frac{n}{2q} - \frac{1}{\ell}$ while r and l are defined by

$$\frac{1}{2s} + \frac{1}{q} + \frac{2-\frac{1}{s}}{r} = 1; \quad \frac{1}{2s} + \frac{1}{\ell} + \frac{2-\frac{1}{s}}{l} = 1.$$

Note that $\frac{n}{2} < \frac{n}{r} + \frac{2}{l} < \frac{n}{2} + 1$, and, by the embedding theorem [LSU, Ch. II, (3.4)],

$$\|v\zeta\|_{r,l,Q_R^{\lambda,\theta}} \leq C_9(n, r, l, \lambda, \theta) R^{\frac{n}{r} + \frac{2}{l} - \frac{n}{2}} \|v\zeta\|_{\mathcal{V}(Q_R^{\lambda,\theta})}. \quad (36)$$

Using (1), (35) and (36) and the Young inequality, we obtain from (34)

$$\|v\zeta\|_{\mathcal{V}(Q_R^{\lambda,\theta})}^2 \leq C_{10}(n, \nu, q, \ell, \lambda, \theta) \cdot \int_{Q_R^{\lambda,\theta}} v^2 (|D\zeta|^2 + \zeta|\partial_t\zeta| + R^2\|\mathbf{b}\|_{q,\ell,Q_R^{\lambda,\theta}}^{2s} \zeta^{2-2s} |D\zeta|^{2s}) dxdt. \quad (37)$$

We put $\lambda_m = 1 + 2^{-m}(\lambda - 1)$, $\theta_m = \frac{\theta}{2}(1 + 4^{-m})$, $m \in \mathbb{N} \cup \{0\}$, and substitute $\zeta = \zeta_m$ such that

$$\zeta_m \equiv 1 \text{ in } Q_R^{\lambda_{m+1}, \theta_{m+1}}; \quad \zeta_m \equiv 0 \text{ out of } Q_R^{\lambda_m, \theta_m}; \quad |\partial_t \zeta_m| \leq \frac{4^m C}{\theta R^2}; \quad \frac{|D\zeta_m|}{\zeta_m^{1-\frac{1}{s}}} \leq \frac{2^m C_3(s)}{(\lambda - 1)R}.$$

Then (37) implies

$$\|v\zeta_m\|_{\mathcal{V}(Q_R^{\lambda_m, \theta_m})} \leq \frac{C_{11}(n, \nu, q, \ell, \lambda, \theta)}{R} \cdot \|v\|_{2,2,Q_R^{\lambda_m, \theta_m}} \cdot (2^m + (2^m \hat{\mathcal{N}})^s). \quad (38)$$

Now for $p = p_m \equiv (\frac{n+2}{n})^m$ we obtain from (36) (with $r = l = \frac{2(n+2)}{n}$) and (38)

$$\begin{aligned} \left(\int_{Q_R^{\lambda_{m+1}, \theta_{m+1}}} u_+^{2p_{m+1}} dxdt \right)^{\frac{1}{2p_{m+1}}} &\leq \left(C(n) \int_{Q_R^{\lambda_m, \theta_m}} (v\zeta_m)^r dxdt \right)^{\frac{1}{rpm}} \leq \\ &\leq \left(2^{2ms} C_{12} \int_{Q_R^{\lambda_m, \theta_m}} v^2 dxdt \right)^{\frac{1}{2pm}} = \left(2^{2ms} C_{12} \int_{Q_R^{\lambda_m, \theta_m}} u_+^{2p_m} dxdt \right)^{\frac{1}{2pm}}, \end{aligned} \quad (39)$$

where C_{12} depends only on $n, \nu, q, \ell, \lambda, \theta$ and $\hat{\mathcal{N}}$.

Iterating (39) we arrive at (32). \square

Remark 5. If $\mathbf{b} = \mathbf{b}^{(1)} + \mathbf{b}^{(2)}$, and $\mathbf{b}^{(j)} \in L_{q_j, \ell_j}(Q_R^{\lambda, \theta}(x^0; t^0))$, $j = 1, 2$, with $\alpha(q_j, \ell_j) \in [0, 1]$, then the proof of Lemma 3.1 does not change. The same is true for other statements of this Section.

Remark 6. If, under assumptions of Lemma 3.1, u satisfies additionally

$$u(\cdot; t^0 - \theta R^2) \leq 0 \quad \text{in } B_{\lambda R}(x^0), \quad (40)$$

then we can estimate u up to the bottom of the cilinder, i.e. one can replace the left-hand side in (32) by $\sup_{Q_R^{1, \frac{\theta}{2}}(x^0; t^0)} u_+$. Indeed, one may simply put $\theta_m \equiv \theta$ and take ζ_m independent on t .

Corollary 3.1. *Let \mathcal{M} satisfy the assumptions of Lemma 3.1 in $Q_R^{\lambda, \theta}(x^0; t^0)$.*

1. If a Lipschitz subsolution of $\mathcal{M}u = 0$ in $Q_R^{\lambda, \theta}(x^0; t^0)$ satisfies

$$\text{meas}(\{u > k\} \cap Q_R^{\lambda, \theta}(x^0; t^0)) \leq \mu \text{meas}(Q_R^{\lambda, \theta}), \quad \mu < N_4^{-2}, \quad (41)$$

for some k , then

$$\sup_{Q_R^{1, \frac{\theta}{2}}(x^0; t^0)} (u - k) \leq N_4 \sqrt{\mu} \sup_{Q_R^{\lambda, \theta}(x^0; t^0)} (u - k), \quad (42)$$

(here N_4 is the constant from Lemma 3.1).

2. If a Lipschitz nonnegative supersolution of $\mathcal{M}V = 0$ in $Q_R^{\lambda, \theta}(x^0; t^0)$ satisfies

$$\text{meas}(\{V < k\} \cap Q_R^{\lambda, \theta}(x^0; t^0)) \leq \mu \text{meas}(Q_R^{\lambda, \theta}), \quad \mu \leq \mu_1 \equiv (2N_4)^{-2}, \quad (43)$$

for some $k > 0$, then

$$V \geq \frac{k}{2} \quad \text{in} \quad Q_R^{1, \frac{\theta}{2}}(x^0; t^0). \quad (44)$$

If V additionally satisfies

$$V(\cdot; t^0 - \theta R^2) \geq k \quad \text{in} \quad B_{\lambda R}(x^0),$$

then the estimate (44) holds in $Q_R^{1, \theta}(x^0; t^0)$.

Proof. 1. We apply Lemma 3.1 to $u - k$.

2. We apply Lemma 3.1 and Remark 6 to $u = k - V$. \square

Lemma 3.2. Let \mathcal{M} satisfy the assumptions of Lemma 3.1 in $Q_R(x^0; t^0)$. For any $\delta_0 \in]0, 1]$ there exists $\theta_0 \in]0, 1[$ such that if a Lipschitz nonnegative supersolution of $\mathcal{M}V = 0$ in $Q_R^{1, \theta_0}(x^0; t^0)$ satisfies

$$\text{meas}(\{V(\cdot; t^0 - \theta_0 R^2) \geq k\} \cap B_R(x^0)) \geq \delta_0 \text{meas}(B_R)$$

for some $k > 0$, then

$$\text{meas}(\{V(\cdot; \bar{t}) \geq \frac{\delta_0}{3}k\} \cap B_R(x^0)) \geq \frac{\delta_0}{3} \text{meas}(B_R) \quad \text{for any} \quad \bar{t} \in [t^0 - \theta_0 R^2, t^0].$$

Moreover, θ_0 is completely determined by δ_0 , n , ν , q , ℓ and the quantity $\widehat{\mathcal{N}}$.

Proof. Without loss of generality, we assume $(x^0; t^0) = (0; 0)$. For a nonnegative test function η we have

$$\int_{Q_R^{\lambda, \theta_0}} (\partial_t V \eta + a_{ij} D_j V D_i \eta + b_i D_i V \eta) dx dt \geq 0. \quad (45)$$

We take $\eta = \chi_{\{t < \bar{t}\}} \cdot (V - k)_- \zeta^2(x)$, where ζ is a smooth cut-off function in B_R , $\bar{t} \in] - \theta_0 R^2, 0]$. Using (1), (2) and the Young inequality, we obtain

$$\begin{aligned} \int_{B_R} (V - k)_-^2 \zeta^2|^{t=\bar{t}} dx + \nu \int_{Q_R^{1, \theta_0}} \chi_{\{t < \bar{t}\}} |D(V - k)_-|^2 \zeta^2 dx dt &\leq \int_{B_R} (V - k)_-^2 \zeta^2|^{t=-\theta_0 R^2} dx + \\ &+ \int_{Q_R^{1, \theta_0}} \chi_{\{t < \bar{t}\}} \left(C_{13}(n, \nu) (V - k)_-^2 |D\zeta|^2 dx dt + 2b_i (V - k)_-^2 \zeta D_i \zeta \right) dx dt. \end{aligned} \quad (46)$$

Now we choose ζ such that $\zeta \equiv 1$ in $B_{(1-\sigma)R}$ and $|D\zeta| \leq \frac{2}{\sigma R}$ where $\sigma < 1$ is a parameter to be chosen later. Observing that $(V - k)_-^2 \leq k^2$, we estimate the right-hand side of (46) by

$$k^2 \left[(1 - \delta_0) \text{meas}(B_R) + C_{13} \theta_0 R^2 \cdot \frac{4 \text{meas}(B_R)}{(\sigma R)^2} + \frac{2}{\sigma R} \|\mathbf{b}\|_{q, \ell, Q_R} \|\mathbf{1}\|_{q', \ell', Q_R^{1, \theta_0}} \right].$$

On the another hand,

$$\int_{B_R} (V-k)_-^2 \zeta^2 |^{t=\bar{t}} dx \geq \int_{\{V < \frac{\delta_0}{3} k\} \cap B_{(1-\sigma)R}} (V-k)_-^2 |^{t=\bar{t}} dx \geq (1-\frac{\delta_0}{3})^2 k^2 \text{meas}(\{V(\cdot; \bar{t}) < \frac{\delta_0}{3} k\} \cap B_{(1-\sigma)R}).$$

Thus,

$$\text{meas}(\{V(\cdot; \bar{t}) < \frac{\delta_0}{3} k\} \cap B_{(1-\sigma)R}) \leq \frac{\text{meas}(B_R)}{(1-\frac{\delta_0}{3})^2} \cdot \left[(1-\delta_0) + \frac{4C_{13}\theta_0}{\sigma^2} + \frac{C(n)\theta_0^{\frac{2}{\ell}} \widehat{\mathcal{N}}}{\sigma} \right],$$

and therefore,

$$\text{meas}(\{V(\cdot; \bar{t}) < \frac{\delta_0}{3} k\} \cap B_R) \leq \frac{\text{meas}(B_R)}{(1-\frac{\delta_0}{3})^2} \cdot \left[(1-\delta_0) + C(n)\sigma + \frac{4C_{13}\theta_0}{\sigma^2} + \frac{C(n)\theta_0^{\frac{2}{\ell}} \widehat{\mathcal{N}}}{\sigma} \right].$$

Since $1-\delta_0 \leq (1-\frac{\delta_0}{3})^3 - \frac{8}{27}\delta_0^2$, one can choose σ and then θ_0 small enough such that the right-hand side is not greater than $(1-\frac{\delta_0}{3}) \text{meas}(B_R)$, and the Lemma follows. \square

Lemma 3.3. *Let \mathcal{M} satisfy the assumptions of Lemma 3.1 in $Q_R^{\lambda, \theta}(x^0; t^0)$ with $\lambda > 1$. Let a Lipschitz nonnegative supersolution of $\mathcal{M}V = 0$ in $Q_R^{\lambda, \theta}(x^0; t^0)$ satisfy*

$$\text{meas}(\{V(\cdot; t) \geq k_0\} \cap B_R(x^0)) \geq \delta_1 \text{meas}(B_R) \quad \text{for any } t \in [t^0 - \theta R^2, t^0] \quad (47)$$

for some $k_0 > 0$ and $\delta_1 > 0$. Then for any $\mu \in]0, 1[$ there exists $s > 1$ such that

$$\text{meas}(\{V < 2^{-s} k_0\} \cap Q_R^{1, \theta}(x^0; t^0)) \leq \mu \text{meas}(Q_R^{1, \theta}).$$

Moreover, s is completely determined by $n, \nu, \lambda, \theta, \mu, \delta_1, q, \ell$, and the quantity $\widehat{\mathcal{N}}$.

Proof. Without loss of generality, we assume $(x^0; t^0) = (0; 0)$. For $m \in \mathbb{Z}_+$ we put $k_m = 2^{-m} k_0$,

$$\mathcal{E}_m(t) = \{x \in B_R : k_{m+1} \leq V(x, t) < k_m\}; \quad \mathcal{E}_m = \{(x; t) : t \in [-\theta R^2, 0], x \in \mathcal{E}_m(t)\}.$$

We take in (45) $\eta = (V - k_m)_- \zeta^2(x)$, where ζ is a smooth cut-off function, vanishing at the neighborhood of $\partial B_{\lambda R}$ and satisfying $\zeta \equiv 1$ in B_R , $|D\zeta| \leq \frac{2}{(\lambda-1)R}$. Similarly to the proof of Lemma 3.2, we derive

$$\int_{\{V < k_m\}} |DV|^2 \zeta^2 dx dt = \int_{Q_R^{\lambda, \theta}} |D(V - k_m)_-|^2 \zeta^2 dx dt \leq C_{14}(n, \nu, \lambda, \theta, \ell, \widehat{\mathcal{N}}) k_m^2 R^n. \quad (48)$$

Further, De Giorgi's inequality (see, e.g., [LSU, Ch. II, (5.6)]) and the assumption (47) give

$$(k_m - k_{m+1}) \cdot \text{meas}(\{V(\cdot; t) < k_{m+1}\} \cap B_R) \leq \frac{C(n)R}{\delta_1} \int_{\mathcal{E}_m(t)} |DV(\cdot; t)| dx, \quad t \in [-\theta R^2, 0].$$

We integrate this relation w.r.t t and then square both parts, arriving at

$$k_{m+1}^2 \text{meas}^2(\{V < k_{m+1}\} \cap Q_R^{1, \theta}) \leq \frac{C(n)R^2}{\delta_1^2} \int_{\mathcal{E}_m} |DV|^2 dx dt \cdot \text{meas}(\mathcal{E}_m).$$

Together with (48), this gives

$$\text{meas}^2(\{V < k_{m+1}\} \cap Q_R^{1,\theta}) \leq C(n)C_{14}\delta_1^{-2}R^{n+2} \cdot \text{meas}(\mathcal{E}_m).$$

Therefore,

$$\begin{aligned} s \cdot \text{meas}^2(\{V < k_s\} \cap Q_R^{1,\theta}) &\leq \sum_{m=0}^{s-1} \text{meas}^2(\{V < k_{m+1}\} \cap Q_R^{1,\theta}) \leq \\ &\leq C_{15}\delta_1^{-2} \cdot \text{meas}(Q_R^{1,\theta}) \cdot \sum_{m=0}^{s-1} \text{meas}(\mathcal{E}_m) \leq C_{15}\delta_1^{-2} \cdot \text{meas}^2(Q_R^{1,\theta}) \end{aligned}$$

(here C_{15} depends on the same quantities as C_{14}), and the Lemma follows. \square

Corollary 3.2. *Let \mathcal{M} satisfy the assumptions of Lemma 3.1 in $Q_R^{2,1}$.*

1. *If a Lipschitz nonnegative supersolution of $MV = 0$ in $Q_R^{2,1}$ satisfies*

$$\text{meas}(\{V(\cdot; \bar{t}) \geq k\} \cap B_R) \geq \delta \text{meas}(B_R) \quad (49)$$

for some $\bar{t} \in [-R^2, -\Theta R^2]$, $k > 0$, $\delta, \Theta \in]0, 1]$, then

$$V \geq \beta_1 k \quad \text{in} \quad Q_R^{1, \frac{\theta_1}{2}}(0; \bar{t} + \theta_1 R^2). \quad (50)$$

Here $\theta_1 = \min\{\Theta, \theta_0\}$, where $\theta_0 = \theta_0(\delta, n, \nu, q, \ell, \widehat{\mathcal{N}})$ is the constant from Lemma 3.2, while β_1 depends on the same quantities as θ_0 .

2. *If the relation (49) holds with $\delta = 1$, i.e.*

$$V(\cdot; \bar{t}) \geq k \quad \text{in} \quad B_R, \quad (51)$$

then for any $\sigma \in]0, 1[$

$$V \geq \beta_1 k \quad \text{in} \quad Q_R^{\sigma, \theta_1}(0; \bar{t} + \theta_1 R^2), \quad (52)$$

In this case β_1 depends additionally on σ .

Proof. First, we use Lemma 3.2 with $t^0 = \bar{t} + \theta_1 R^2$. Then, in the case 1, we apply Lemma 3.3 with $R \rightarrow \frac{3}{2}R$, $\lambda = \frac{4}{3}$, $\theta = \frac{4}{9}\theta_1$, $\delta_1 = \frac{2^n \delta}{3^{n+1}}$, $k_0 = \frac{\delta}{3}k$ and $\mu = \mu_1$, where $\mu_1 = \mu_1(n, \nu, \frac{3}{2}, \theta_1, q, \ell, \widehat{\mathcal{N}})$ is the constant from Corollary 3.1, part 2. Finally, Corollary 3.1 with $\lambda = \frac{3}{2}$ and $\theta = \theta_1$ gives (50) with $\beta_1 = \frac{\delta}{3 \cdot 2^{s+1}}$, where $s = s(n, \nu, \frac{3}{2}, \theta_1, \mu_1, \delta_1, q, \ell, \widehat{\mathcal{N}})$ is the constant from Lemma 3.3.

In the case 2, we apply Lemma 3.3 with $\lambda = 2$, $\theta = \theta_1$, $\delta_1 = \frac{\delta}{3}$, $k_0 = \frac{\delta}{3}k$ and $\mu = \mu_1(n, \nu, \sigma^{-1}, \sigma^{-2}\theta_1, q, \ell, \widehat{\mathcal{N}})$. Finally, the last statement of Corollary 3.1 with $R \rightarrow \sigma R$, $\lambda = \sigma^{-1}$ and $\theta = \sigma^{-2}\theta_1$ gives (52) with $\beta_1 = \frac{\delta}{3 \cdot 2^{s+1}}$, where $s = s(n, \nu, \sigma^{-1}, \sigma^{-2}\theta_1, \mu_1, \delta_1, q, \ell, \widehat{\mathcal{N}})$. \square

Lemma 3.4. *Let \mathcal{M} satisfy the assumptions of Lemma 3.1 in $Q_R^{2,1}$. Let a Lipschitz non-negative supersolution of $MV = 0$ in $Q_R^{2,1}$ satisfy (51) for some $k > 0$ and $\bar{t} \in [-R^2, -\Theta R^2]$, $\Theta \in]0, 1]$. Then*

$$V \geq \beta_2 k \quad \text{in} \quad \widehat{Q} = B_{\frac{R}{2}} \times [\bar{t}, 0]. \quad (53)$$

Moreover, β_2 is completely determined by Θ , n , ν , q , ℓ , and the quantity $\widehat{\mathcal{N}}$.

Proof. We set $M = \text{entier}\left(\frac{|\bar{t}|}{\theta_1 R^2}\right) + 1$ and $\widehat{\theta}_1 = \frac{|\bar{t}|}{MR^2}$. Now let us consider cylinders

$$Q^{(m)} = Q_R^{1-\frac{m}{2M}, \widehat{\theta}_1}(0; \bar{t} + m\widehat{\theta}_1 R^2), \quad m = 1, \dots, M.$$

By (52) we consequently obtain

$$V \geq \widehat{\beta}_1 \cdot \inf_{Q^{(m)}} V \quad \text{in } Q^{(m+1)},$$

where $\widehat{\beta}_1 = \widehat{\beta}_1(n, \Theta, \nu, q, \ell, \widehat{\mathcal{N}}) > 0$.

Since $\widehat{Q} \subset \bigcup_m Q^{(m)}$, this ensures (53) with $\beta_2 = \widehat{\beta}_1^M$. \square

Corollary 3.3. *Let \mathcal{M} satisfy the assumptions of Lemma 3.1 in $Q_R^{2,1}$. If a Lipschitz nonnegative supersolution of $\mathcal{M}V = 0$ in $Q_R^{2,1}$ satisfies (49) for some $k > 0$, $\delta \in]0, 1]$ and $\bar{t} \in [-R^2, -\frac{3}{4}R^2]$, then*

$$V \geq \beta_3 k \quad \text{in } Q_{\frac{R}{2}}, \quad (54)$$

where β_3 is completely determined by δ , n , ν , q , ℓ , and the quantity $\widehat{\mathcal{N}}$.

Proof. It suffices to apply consequently Corollary 3.2, part 1, and Lemma 3.4. \square

Corollary 3.4. *Let \mathcal{M} satisfy the assumptions of Lemma 3.1 in $Q_R^{2,1}$. If a Lipschitz nonnegative supersolution of $\mathcal{M}V = 0$ in $Q_R^{2,1}$ satisfies*

$$\text{meas}(\{V > k\} \cap Q_R) \geq \widehat{\delta} \text{meas}(Q_R), \quad (55)$$

for some $k > 0$, $\widehat{\delta} \in]0, 1]$, then

$$V \geq \beta_4 k \quad \text{in } Q_{\frac{R}{2}}^{\frac{1}{2}, \widehat{\delta}}, \quad (56)$$

where β_4 is completely determined by $\widehat{\delta}$, n , ν , q , ℓ , and the quantity $\widehat{\mathcal{N}}$.

Proof. The inequality (55) obviously implies

$$\text{meas}(\{V > k\} \cap Q_R^{1,1-\frac{\widehat{\delta}}{2}}(0; -\frac{\widehat{\delta}}{2}R^2)) \geq \frac{\widehat{\delta}}{2} \text{meas}(Q_R).$$

Therefore, there exists $\bar{t} \in [-R^2, -\frac{\widehat{\delta}}{2}R^2]$, such that (49) holds with $\delta = \frac{\widehat{\delta}}{2}$. By Corollary 3.2, part 1, $V(\cdot; \bar{t} + \frac{\theta_1}{2}R^2) \geq \beta_1 k$ in B_R . Finally, we observe that $\bar{t} + \frac{\theta_1}{2}R^2 \leq -\frac{\widehat{\delta}}{4}R^2$, and Lemma 3.4 provides (56). \square

Corollary 3.5 (strong maximum principle). *Let \mathcal{M} satisfy the assumption of Lemma 3.1 in Q . Then any Lipschitz nonconstant supersolution of $\mathcal{M}V = 0$ in Q cannot attain its minimum at a point of $\partial Q \setminus \partial' Q$.*

Proof. Without loss of generality, $\inf_Q V = 0$.

Assume the converse. Then there exists $(x^0; t^0) \in \overline{Q} \setminus \partial' Q$ such that $V(x^0; t^0) = 0$ but $V \not\equiv 0$ in $Q_R(x^0; t^0) \subset Q_R^{2,1}(x^0; t^0) \subset Q$ with some R . Then the relation (55) holds for some $k > 0$ and $\delta > 0$, and we obtain (56), a contradiction. \square

Lemma 3.5. *Let \mathcal{M} satisfy the assumptions of Lemma 3.1 in Q_{2R} . Then any Lipschitz solution of $\mathcal{M}u = 0$ in Q_{2R} satisfies the estimate*

$$\operatorname{osc}_{Q_{\frac{R}{2}}} u \leq \varkappa_1 \operatorname{osc}_{Q_{2R}} u, \quad (57)$$

where $\varkappa_1 < 1$ depends on n, ν, q, ℓ and the quantity $\widehat{\mathcal{N}}$.

Proof. We set $k = \frac{1}{2} \operatorname{osc}_{Q_{2R}} u$ and consider two functions $V_1 = u - \inf_{Q_{2R}} u$ and $V_2 = \sup_{Q_{2R}} u - u$. At least one of them satisfies (49) with $\delta = \frac{1}{2}$ and $\bar{t} = -R^2$. Therefore, Corollary 3.3 gives for this function the estimate (54), which implies (57) with $\varkappa_1 = 1 - \frac{1}{2} \beta_3(\frac{1}{2}, n, \nu, q, \ell, \widehat{\mathcal{N}})$. \square

Corollary 3.6 (Hölder estimate). *Let \mathcal{M} satisfy the assumption of Lemma 3.1 in Q_{R_0} . Let also $\sup_{R < R_0} \widehat{\mathcal{N}}(R, 1, 1) < \infty$. Then any Lipschitz solution of $\mathcal{M}u = 0$ in Q_{R_0} satisfies the estimate*

$$\operatorname{osc}_{Q_\rho} u \leq N_5 \left(\frac{\rho}{r} \right)^{\gamma_1} \cdot \operatorname{osc}_{Q_r} u, \quad 0 < \rho < r \leq R_0, \quad (58)$$

where N_5 and γ_1 depend on n, ν, q and $\sup_{R < R_0} \widehat{\mathcal{N}}(R, 1, 1)$.

Proof. Iteration of (57) gives (58) with $\gamma_1 = -\log_4(\varkappa_1)$. \square

Corollary 3.7 (the Liouville theorem). *Let \mathcal{M} be an operator of the form **(DP)** in $\mathbb{R}^n \times \mathbb{R}_-$, and let the conditions (1) and (2) be satisfied. Let also $\mathbf{b} \in L_{q, \ell, \text{loc}}(\mathbb{R}^n \times \mathbb{R}_-)$, with some q and ℓ satisfying (31) (q as well as ℓ may be infinite). Finally, assume that*

$$\liminf_{R \rightarrow \infty} \widehat{\mathcal{N}}(R, 1, 1) < \infty. \quad (59)$$

Then any Lipschitz bounded solution of $\mathcal{M}u = 0$ in $\mathbb{R}^n \times \mathbb{R}_-$ is a constant.

Remark 7. If $\mathbf{b} \in L_{q, \ell}(\mathbb{R}^n \times \mathbb{R}_-)$, then (59) is obviously satisfied.

Proof. Iteration of (57) with respect to a suitable sequence $R_m \rightarrow \infty$ gives the statement. \square

To prove the Harnack inequality, we need the following modification of Lemma 3.4.

Lemma 3.4' (slant cylinder). *Let \mathcal{M} satisfy the assumptions of Lemma 3.1 in $Q_R^{\lambda, 1}$, $\lambda > 2$. Let a Lipschitz nonnegative supersolution of $\mathcal{M}V = 0$ in $Q_R^{\lambda, 1}$ satisfy*

$$V(\cdot; -R^2) \geq k \quad \text{in } B_R(x^0),$$

for some $k > 0$ and $x^0 \in B_{(\lambda-2)R}$. Then for any $x^1 \in B_{(\lambda-2)R}$

$$V \geq \widehat{\beta}_2 k \quad \text{in } \widetilde{Q} = \{(x; t) : t \in [-R^2, 0], x \in B_{\frac{R}{2}}(x^1 + (x^1 - x^0) \frac{t}{R^2})\}. \quad (60)$$

Moreover, $\widehat{\beta}_2$ is completely determined by λ, n, ν, q, ℓ , and the quantity $\widehat{\mathcal{N}}$.

Proof. We put $\widehat{x}(t) = x^1 + (x^1 - x^0) \frac{t}{R^2}$. Then it is easy to see that the function $\widetilde{V}(x; t) = V(x - \widehat{x}(t); t)$ is a Lipschitz nonnegative supersolution of

$$\widetilde{\mathcal{M}}V \equiv \partial_t V - D_i(\widetilde{a}_{ij}(x; t) D_j V) + \widetilde{b}_i(x; t) D_i V = 0$$

in $Q_R^{2,1}$, where

$$\tilde{a}_{ij}(x; t) = a_{ij}(x - \hat{x}(t); t); \quad \tilde{b}_i(x; t) = b_i(x - \hat{x}(t); t) + \frac{x_i^1 - x_i^0}{R^2}.$$

Note that $\tilde{\mathcal{M}}$ satisfies the assumptions of Lemma 3.1 in $Q_R^{2,1}$, and the quantity $\tilde{\mathcal{N}}$ is bounded by $\hat{\mathcal{N}} + C_{16}(\lambda, n)$. By Lemma 3.4 (with $\Theta = 1$), we obtain (60). \square

The next statement is a parabolic analog of Lemma 2.4. For $(x; t) \in Q$ we introduce the notation $d_{\text{par}}((x; t), \partial' Q) = \inf\{\rho > 0 : Q_\rho(x; t) \subset Q\}$.

Lemma 3.6. *Let \mathcal{M} satisfy the assumptions of Lemma 3.1 in Q_{2R} , and let $\mathbf{b} \in \mathbb{M}_{q, \ell}^\alpha(Q_{2R})$. Let for a Lipschitz nonnegative supersolution of $\mathcal{M}V = 0$ in Q_{2R} and some $(y; s) \in Q_R^{2,2}(0; -2R^2)$, the inequality $\inf_{B_\rho(y)} V(\cdot; s) = k > 0$ holds with $\rho = \frac{1}{4}d_{\text{par}}((y; s), \partial' Q_{2R})$. Then*

$$\inf_{Q_R} V \geq N_6 \left(\frac{\rho}{R} \right)^{\hat{\gamma}_1} k, \quad (61)$$

where N_6 and $\hat{\gamma}_1$ depend on n, ν, q, ℓ and $\|\mathbf{b}\|_{\mathbb{M}_{q, \ell}^\alpha(Q_{2R})}$.

Proof. We denote by \mathfrak{N} an integer number such that $2^{-(\mathfrak{N}+1)}R \leq \rho < 2^{-\mathfrak{N}}R$ and introduce a sequence of cylinders $Q_{\mathfrak{r}_m}^{4,1}(y^m; t^m)$, $m = 0, \dots, \mathfrak{N}$, as follows:

$$\begin{aligned} \mathfrak{r}_0 &= 2^{-(\mathfrak{N}+1)}R, & y^0 &= y, & t^0 &= s + \mathfrak{r}_0^2; \\ \mathfrak{r}_m &= 2\mathfrak{r}_{m-1}, & y^m &= y^{m-1} - \min\{2\mathfrak{r}_m, |y^{m-1}|\} \mathbf{e}, & t^m &= t^{m-1} + \mathfrak{r}_m^2 \end{aligned}$$

(here $\mathbf{e} = \frac{y}{|y|}$). Also we denote $y^{-1} = y$, $t^{-1} = s$.

Direct computation shows that $Q_{\mathfrak{r}_m}^{4,1}(y^m; t^m) \subset Q_{2R}$ for all $m = 0, \dots, \mathfrak{N}$. Therefore, the assumptions of Lemma 3.4' (with $\lambda = 4$, $x^0 = y^{m-1}$) are fulfilled in $Q_{\mathfrak{r}_m}^{4,1}(y^m; t^m)$. Using Lemma 3.4' with $x^1 \in B_{2\mathfrak{r}_m}(y^m)$, we obtain the inequality

$$V(\cdot; t^m) \geq \hat{\beta}_2(4, n, \nu, q, \ell, \|\mathbf{b}\|_{\mathbb{M}_{q, \ell}^\alpha(Q_{2R})}) \cdot \inf_{B_{\mathfrak{r}_m}(y^{m-1})} V(\cdot; t^{m-1}) \quad \text{in } B_{\frac{5}{2}\mathfrak{r}_m}(y^m)$$

and, in particular,

$$\inf_{B_{\mathfrak{r}_{m+1}}(y^m)} V(\cdot; t^m) \geq \hat{\beta}_2 \cdot \inf_{B_{\mathfrak{r}_m}(y^{m-1})} V(\cdot; t^{m-1}).$$

Since $\mathfrak{r}_{\mathfrak{N}+1} = R$ and $y^{\mathfrak{N}} = 0$, we obtain

$$\inf_{B_R} V(\cdot; t^{\mathfrak{N}}) \geq \hat{\beta}_2^{\mathfrak{N}+1} \cdot \inf_{B_{\mathfrak{r}_0}(y)} V(\cdot; s) \geq \left(\frac{\rho}{2R} \right)^{\hat{\gamma}_1} \cdot k,$$

where $\hat{\gamma}_1 = -\log_2(\hat{\beta}_2)$.

It is easy to estimate $t^{\mathfrak{N}}$:

$$t^{\mathfrak{N}} = s + \sum_{m=0}^{\mathfrak{N}} \mathfrak{r}_m^2 = s + \mathfrak{r}_0^2 \cdot \frac{2^{2\mathfrak{N}+2} - 1}{3} < s + \frac{R^2}{3} \leq -\frac{5}{3}R^2.$$

Now we use Lemma 3.4' (with $\lambda = 4$, $x^0 \in B_{\frac{R}{2}}$, $x^1 \in B_R$) in $Q_{\frac{R}{2}}^{4,1}(0; t^{\mathfrak{N}} + \frac{R^2}{4})$. Since slant cylinders for $x^0 \in B_{\frac{R}{2}}$, $x^1 \in B_R$ cover $Q_R^{1, \frac{1}{8}}(0; t^{\mathfrak{N}} + \frac{R^2}{4})$, we obtain

$$\inf_{Q_R^{1, \frac{1}{8}}} (0; t^{\mathfrak{N}} + \frac{R^2}{4}) V \geq \hat{\beta}_2 \left(\frac{\rho}{2R} \right)^{\hat{\gamma}_1} \cdot k.$$

Repeating this process, we cover Q_R . This gives (61) with $N_6 = 2^{-\hat{\gamma}_1} \hat{\beta}_2^{31}$. \square

Theorem 3.7 (the Harnack inequality). *Let \mathcal{M} be an operator of the form (DP) in Q_{2R} , and let the conditions (1) and (2) be satisfied. Let also $\mathbf{b} \in \mathbb{M}_{q,\ell}^\alpha(Q_{2R})$, with some q and ℓ satisfying (31) (as q as ℓ may be infinite).*

Then there exists a positive constant N_7 depending on n, ν, q, ℓ and $\|\mathbf{b}\|_{\mathbb{M}_{q,\ell}^\alpha(Q_{2R})}$, such that any Lipschitz nonnegative solution of $\mathcal{M}u = 0$ in Q_{2R} satisfies

$$\sup_{Q_R(0;-2R^2)} u \leq N_7 \cdot \inf_{Q_R} u. \quad (62)$$

Proof. Similarly to Theorem 2.5, we denote by $(y; s)$ a maximum point of the function

$$v(x; t) = (d_{\text{par}}((x; t), \partial' Q_{2R}))^{\hat{\gamma}_1} \cdot u(x; t); \quad (x; t) \in Q_R^{2,2}(0; -2R^2)$$

(here $\hat{\gamma}_1$ is the constant from Lemma 3.6) and set

$$\rho = \frac{1}{4} d_{\text{par}}((y; s), \partial' Q_{2R}); \quad \mathfrak{M} = v(y; s) = (4\rho)^{\hat{\gamma}_1} \cdot u(y; s).$$

It is obvious that

$$\sup_{Q_R(0;-2R^2)} u \leq \frac{\mathfrak{M}}{R^{\hat{\gamma}_1}} = \left(\frac{4\rho}{R}\right)^{\hat{\gamma}_1} \cdot u(y; s); \quad (63)$$

$$\sup_{Q_{2\rho}(y;s)} u \leq \frac{\mathfrak{M}}{(2\rho)^{\hat{\gamma}_1}} = 2^{\hat{\gamma}_1} \cdot u(y; s). \quad (64)$$

Denote $k = \frac{1}{2}u(y; s)$. If $\text{meas}(\{u > k\} \cap Q_{2\rho}(y; s)) \leq \mu \text{meas}(Q_{2\rho})$, then Corollary 3.1, part 1 (with $\lambda = 2$ and $\theta = 4$) and (64) imply the relation

$$k = u(y; s) - k \leq \sup_{Q_{2\rho}(y;s)} (u - k) \leq N_4 \sqrt{\mu} \sup_{Q_{2\rho}(y;s)} (u - k) \leq N_4 \sqrt{\mu} (2^{\hat{\gamma}_1+1} - 1)k,$$

which is impossible for $\mu \leq \mu_2 \equiv \frac{1}{2^{2\hat{\gamma}_1+2}} N_4^{-2}$. Thus, $\text{meas}(\{u > k\} \cap Q_{2\rho}(y; s)) \geq \mu_2 \text{meas}(Q_{2\rho})$, and Corollary 3.4 (with $\hat{\delta} = \mu_2$) gives

$$\inf_{B_\rho(y)} u(\cdot; s) \geq \beta_4 k = \frac{\beta_4}{2} \cdot u(y; s). \quad (65)$$

Finally, Lemma 3.6 gives

$$\inf_{Q_R} u \geq N_6 \left(\frac{\rho}{R}\right)^{\hat{\gamma}_1} \inf_{B_\rho(y)} u(\cdot; s). \quad (66)$$

Combining (63), (65) and (66), we arrive at (62) with $N_7 = \frac{2^{2\hat{\gamma}_1+1}}{\beta_4 N_6}$. \square

As in Section 2, the proofs of Lemmas 3.1–3.3 run without changes also for weak sub/supersolutions of $\mathcal{M}u = 0$ if the bilinear form

$$\widehat{\mathcal{B}}\langle u, \eta \rangle \equiv \int_{Q_R^{\lambda,\theta}} b_i D_i u \eta \, dx dt$$

can be continuously extended to the pair $(v, v\zeta^2)$ with $v \in \mathcal{V}(Q_R^{\lambda,\theta})$. Unfortunately, we have no parabolic analog of sharp results by Maz'ya–Verbitsky, so we can give only rather rough sufficient conditions. The simplest one is

$$\mathbf{b} \in \mathbb{M}_{q,\infty}^{\frac{n}{q}-1}(Q_R^{\lambda,\theta}), \quad \frac{n}{2} < q \leq n, \quad \text{div}(\mathbf{b}) = 0.$$

If Lemmas 3.1–3.3 are proved, all subsequent statements obviously hold true.

In addition, let us consider the case $\alpha(q, \ell) = 0$, i.e. $\frac{n}{q} + \frac{2}{\ell} = 1$. As in the elliptic case, main results of this section hold true for weak (sub/super)solutions without the assumption (2)⁵.

⁵Also the assumptions on \mathbf{b} can be weakened in the scale of Lorentz spaces.

The only exceptional situation is $q = n$, where the assumption (2) seems to be unavoidable without the smallness restriction on \mathbf{b} ⁶. We explain briefly changes in the proofs.

Similarly to Lemma 2.1, Lemma 3.1 in this case can be proved under additional assumption $\|\mathbf{b}\|_{q,\ell,Q_R^{\lambda,\theta}} \leq \varepsilon(n, \nu)$.

Lemmas 3.2 and 3.3 are proved without changes. Therefore, all subsequent statements hold true under assumption of sufficient smallness of \mathbf{b} .

In what follows we will assume $q > n$. Then, as in the elliptic case, strong maximum principle holds without smallness assumption on \mathbf{b} .

When proving the Harnack inequality, one can exclude the smallness assumption on \mathbf{b} similarly to Theorem 2.5'. The result reads as follows.

Theorem 3.7' (the Harnack inequality). *Let \mathcal{M} be an operator of the form (DE) in Q_{2R} , and let the condition (1) be satisfied. Suppose also that*

$$\|\mathbf{b}\|_{q,\ell,Q_{2R}} \leq \mathfrak{B}, \quad \frac{n}{q} + \frac{2}{\ell} = 1, \quad q > n.$$

Then there exists a positive constant N'_7 depending only on n, ν, q and \mathfrak{B} such that any nonnegative weak solution of $\mathcal{M}u = 0$ in Q_{2R} satisfies

$$\sup_{Q_R(0;-2R^2)} u \leq N'_7 \cdot \inf_{Q_R} u.$$

As in the elliptic case, we have two possibilities to prove the Hölder estimates for solutions. The first one is to take in Lemma 3.5 R sufficiently small, such that the smallness assumptions on \mathbf{b} are satisfied in Q_{2R} . This gives the estimate (58) with γ_1 depending only on n, ν and q , while N_5 depends also on the moduli of integral continuity of \mathbf{b} in $L_{q,\ell}(Q_{R_0})$. The second possibility is to use Theorem 3.7'. This gives (58) with both γ_1 and N_5 depending on n, ν, q and $\|\mathbf{b}\|_{q,\ell,Q_{R_0}}$.

Finally, the next statement directly follows from the second variant of the Hölder estimate.

Corollary 3.7' (the Liouville theorem). *Let \mathcal{M} be an operator of the form (DP) in $\mathbb{R}^n \times \mathbb{R}_-$, and let the conditions (1) be satisfied. Let also $\mathbf{b} \in L_{q,\ell}(\mathbb{R}^n \times \mathbb{R}_-)$, with some q and ℓ such that $\frac{n}{q} + \frac{2}{\ell} = 1, q > n$. Then any weak bounded solution of $\mathcal{M}u = 0$ in $\mathbb{R}^n \times \mathbb{R}_-$ is a constant.*

4 Application to a problem of hydrodynamics

When considering axisymmetric flows of viscous incompressible liquid, the following equation of (DP) form arises:

$$\mathcal{M}u \equiv \partial_t u - \Delta u + b_i(x', x_3; t) D_i u = 0 \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_-. \quad (67)$$

Here we denote $x' = (x_1, x_2)$;

$$\mathbf{b} = \mathbf{v} + \widehat{\mathbf{b}} = \left(v^1 + \varepsilon \frac{2x_1}{|x'|^2}, v^2 + \varepsilon \frac{2x_2}{|x'|^2}, v^3 \right), \quad (68)$$

where $\mathbf{v} = (v^1, v^2, v^3)$ is a solution to the Navier–Stokes system (NSE) while $\varepsilon = \pm 1$.

⁶By Remark 4, if the assumption (2) holds, the case $q = n, \ell = \infty$ is simply included into the case $q = n, 2 < \ell < \infty$.

Namely, see [KNSS], the function $u = v_2x_1 - v_1x_2 \equiv |x'|v_\vartheta$ satisfies the equation (67) with $\varepsilon = +1$ (here v_ϑ is the angular component of the velocity). Next, if $v_\vartheta = 0$, then the function $u = |x'|^{-2}((\text{rot}(\mathbf{v}))_2x_1 - (\text{rot}(\mathbf{v}))_1x_2)$ satisfies the equation (67) with $\varepsilon = -1$.

Since, by the NSE, $\text{div}(\mathbf{v}) = 0$, it is easy to see that

$$\text{div}(\mathbf{b}) = 4\pi\varepsilon\delta_\Gamma, \quad \Gamma = \{|x'| = 0\}. \quad (69)$$

Thus, if $\varepsilon = -1$, then the results of Section 3 are applicable to (67)–(68). Namely, we are interested in the Liouville theorem.

Note that $\widehat{\mathbf{b}} \in L_{q,\infty,\text{loc}}(\mathbb{R}^3 \times \mathbb{R}_-)$ with any $q < 2$, and, moreover, satisfies the assumption (59) with $q \in]\frac{3}{2}, 2[$, $\ell = \infty$. Therefore, taking into account Remark 5, we obtain the following result.

Theorem 4.1. *Let \mathbf{v} be an axisymmetric solution of the Navier–Stokes system in $\mathbb{R}^3 \times \mathbb{R}_-$. Suppose also that \mathbf{v} satisfies (59) with some q and ℓ such that $\alpha \equiv \frac{3}{q} + \frac{2}{\ell} - 1 \in [0, 1[$. Then any Lipschitz bounded solution of (67)–(68) with $\varepsilon = -1$ in $\mathbb{R}^n \times \mathbb{R}_-$ is a constant.*

Remark 8. The assumption (59) is satisfied, for example, if \mathbf{v} satisfies the estimate

$$|\mathbf{v}(x', x_3; t)| \leq \frac{C}{|x'|}$$

(in this case one can take $q \in]\frac{3}{2}, 2[$, $\ell = \infty$), or

$$|\mathbf{v}(x', x_3; t)| \leq \frac{C}{(-t)^{\frac{1}{2}}}$$

(in this case it suffices $q = \infty$, $\ell \in]1, 2[$).

To deal with more complicated case $\varepsilon = +1$, we need the following observation.

Remark 9. The statement of Lemma 3.1 holds true without assumption (2) if $u \leq 0$ in the set $\mathcal{F} = \text{supp}(\text{div}(\mathbf{b}))_+$. Similarly, Lemmas 3.2–3.4 and Corollaries 3.1 (part 2), 3.2, 3.4 hold true if $V \geq k$ in \mathcal{F} . Lemma 3.3 holds true if $V \geq k_0$ in \mathcal{F} .

Now we prove the following variant of Corollary 3.3.

Lemma 4.2. *Let \mathbf{v} be an axisymmetric solution of the Navier–Stokes system in $Q_R^{2,1}$, and let $\mathbf{v} \in L_{q,\ell}(Q_R^{2,1})$ with some q and ℓ such that $\alpha \equiv \frac{3}{q} + \frac{2}{\ell} - 1 \in [0, 1[$. Let V be a Lipschitz nonnegative supersolution of (67)–(68) with $\varepsilon = +1$ in $Q_R^{2,1}$. If*

$$V|_{\Gamma \cap Q_R^{2,1}} \geq k, \quad V \leq \mathfrak{N}k \quad \text{in } Q_R^{2,1} \quad (70)$$

for some $k > 0$ and $\mathfrak{N} > 1$, then

$$V \geq \widehat{\beta}_3 k \quad \text{in } Q_{\frac{R}{2}}, \quad (71)$$

where $\widehat{\beta}_3$ is completely determined by q , ℓ , \mathfrak{N} and the quantity $\widehat{\mathcal{N}} = R^{-\alpha} \|\mathbf{v}\|_{q,\ell,Q_R^{2,1}}$.

Proof. We put

$$\widehat{\mathcal{E}}_\varkappa(t) = \{x \in B_R : V(x, t) > \varkappa k\}; \quad \widehat{\mathcal{E}}_\varkappa = \{(x; t) : t \in [-R^2, -\frac{3}{4}R^2], x \in \widehat{\mathcal{E}}_\varkappa(t)\}$$

and claim that

$$\text{meas}(\widehat{\mathcal{E}}_\varkappa) \geq \delta \text{meas}(Q_R^{1,\frac{1}{4}}) \quad (72)$$

for some $\varkappa > 0$ and $\delta > 0$ depending only on q, ℓ, \mathfrak{N} and $\widehat{\mathcal{N}}$.

Indeed⁷, by (69), we obtain for any Lipschitz test function $\eta \geq 0$

$$\int_{Q_R^{1, \frac{1}{4}}(0; -\frac{3}{4}R^2)} (\partial_t V \eta + D_i V D_i \eta - b_i V D_i \eta) dx dt \geq 4\pi \int_{\Gamma \cap Q_R^{1, \frac{1}{4}}(0; -\frac{3}{4}R^2)} V \eta dx_3 dt. \quad (73)$$

We take η such that

$$\eta \equiv 1 \text{ in } Q_R^{\frac{1}{2}, \frac{1}{8}}(0; -\frac{13}{16}R^2); \quad \eta \equiv 0 \text{ out of } Q_R^{1, \frac{1}{4}}(0; -\frac{3}{4}R^2); \quad |\partial_t \eta| + |D\eta|^2 + |\Delta \eta| \leq \frac{C}{R^2}.$$

Then (73) and $V|_{\Gamma \cap Q_R^{2,1}} \geq k$ imply

$$\begin{aligned} \frac{\pi}{2} k R^3 &\leq \int_{Q_R^{1, \frac{1}{4}}(0; -\frac{3}{4}R^2)} V (|\partial_t \eta| + |\Delta \eta| + |\mathbf{b}| \cdot |D\eta|) dx dt \leq \\ &\leq \frac{C}{R^2} \int_{Q_R^{1, \frac{1}{4}}(0; -\frac{3}{4}R^2)} V dx dt + \frac{C}{R} \int_{Q_R^{1, \frac{1}{4}}(0; -\frac{3}{4}R^2)} V |\mathbf{v}| dx dt + \frac{C}{R} \int_{Q_R^{1, \frac{1}{4}}(0; -\frac{3}{4}R^2)} \frac{V}{|x'|} dx dt. \end{aligned}$$

Splitting the integrals in the right-hand side into integrals over $\widehat{\mathcal{E}}_\varkappa$ and over its complement, we obtain with regard to $V \leq \mathfrak{N}k$ and to $\frac{1}{|x'|} \in L_{\frac{9}{5}, \infty}(Q_R^{1, \frac{1}{4}}(0; -\frac{3}{4}R^2))$

$$\begin{aligned} \frac{\pi}{2} k R^3 &\leq \frac{C\mathfrak{N}k}{R^2} \left[\text{meas}(\widehat{\mathcal{E}}_\varkappa) + R^{1+\alpha} \widehat{\mathcal{N}} \|\mathbf{1}\|_{q', \ell', \widehat{\mathcal{E}}_\varkappa} + R^{\frac{5}{3}} \|\mathbf{1}\|_{\frac{9}{4}, 1, \widehat{\mathcal{E}}_\varkappa} \right] + \\ &+ \frac{C\kappa k}{R^2} \left[\text{meas}(Q_R^{1, \frac{1}{4}}) + R^{1+\alpha} \widehat{\mathcal{N}} \|\mathbf{1}\|_{q', \ell', Q_R^{1, \frac{1}{4}}} + R^{\frac{5}{3}} \|\mathbf{1}\|_{\frac{9}{4}, 1, Q_R^{1, \frac{1}{4}}} \right]. \end{aligned}$$

The second term in the right-hand side is easily estimated by $C\kappa k R^3 (1 + C_{17}(q, \ell) \widehat{\mathcal{N}})$. Therefore, choosing $\varkappa = \varkappa(q, \ell, \widehat{\mathcal{N}})$ sufficiently small, we obtain

$$\frac{C\mathfrak{N}}{R^5} \left[\text{meas}(\widehat{\mathcal{E}}_\varkappa) + R^{1+\alpha} \widehat{\mathcal{N}} \|\mathbf{1}\|_{q', \ell', \widehat{\mathcal{E}}_\varkappa} + R^{\frac{5}{3}} \|\mathbf{1}\|_{\frac{9}{4}, 1, \widehat{\mathcal{E}}_\varkappa} \right] \geq 1. \quad (74)$$

To estimate the second term in brackets, we rewrite it as follows:

$$\|\mathbf{1}\|_{q', \ell', \widehat{\mathcal{E}}_\varkappa} = \left(\int_{-R^2}^{-\frac{3}{4}R^2} \text{meas}^{\frac{\ell'}{q'}}(\widehat{\mathcal{E}}_\varkappa(t)) dt \right)^{\frac{1}{\ell'}}.$$

If $q \leq \ell$ then, by the Hölder inequality,

$$\|\mathbf{1}\|_{q', \ell', \widehat{\mathcal{E}}_\varkappa} \leq \left(\int_{-R^2}^{-\frac{3}{4}R^2} \text{meas}(\widehat{\mathcal{E}}_\varkappa(t)) dt \right)^{\frac{1}{q'}} \cdot \left(\frac{1}{4} R^2 \right)^{\frac{1}{\ell'} - \frac{1}{q'}} = \text{meas}^{\frac{1}{q'}}(\widehat{\mathcal{E}}_\varkappa) \cdot \left(\frac{1}{4} R^2 \right)^{\frac{1}{q} - \frac{1}{\ell}}.$$

⁷This idea was in a particular case used in [CSTY].

In the opposite case, since $\text{meas}(\widehat{\mathcal{E}}_\varkappa(t)) \leq \text{meas}(B_R)$, we obtain

$$\begin{aligned} \|\mathbf{1}\|_{q', \ell', \widehat{\mathcal{E}}_\varkappa} &= \left(\int_{-R^2}^{-\frac{3}{4}R^2} \left(\frac{\text{meas}(\widehat{\mathcal{E}}_\varkappa(t))}{\text{meas}(B_R)} \right)^{\frac{\ell'}{q}} dt \right)^{\frac{1}{\ell'}} \cdot \text{meas}^{\frac{1}{q'}}(B_R) \leq \\ &\leq \left(\int_{-R^2}^{-\frac{3}{4}R^2} \frac{\text{meas}(\widehat{\mathcal{E}}_\varkappa(t))}{\text{meas}(B_R)} dt \right)^{\frac{1}{\ell'}} \cdot \text{meas}^{\frac{1}{q'}}(B_R) = \text{meas}^{\frac{1}{\ell'}}(\widehat{\mathcal{E}}_\varkappa) \cdot (4\pi R^3)^{\frac{1}{\ell} - \frac{1}{q}}. \end{aligned}$$

Similarly we estimate the third term in brackets in (74). Thus, we obtain the inequality for $\mathcal{A} = \frac{1}{R^5} \text{meas}(\widehat{\mathcal{E}}_\varkappa)$:

$$\mathcal{A} + \mathcal{A}^{\frac{4}{9}} + \widehat{\mathcal{N}} \mathcal{A}^{1 - \max\{\frac{1}{q}, \frac{1}{\ell}\}} \geq \frac{1}{C\mathfrak{N}},$$

and (72) follows.

The inequality (72) provides

$$\text{meas}(\widehat{\mathcal{E}}_\varkappa(\bar{t})) \geq \delta \text{meas}(B_R)$$

for some $\bar{t} \in [-R^2, -\frac{3}{4}R^2]$, and Corollary 3.3 ensures (71) with $\widehat{\beta}_3 = \varkappa \cdot \beta_3(\delta, 3, 1, q, \ell, \widehat{\mathcal{N}})$. \square

Theorem 4.3. *Let \mathbf{v} be an axisymmetric solution of the Navier–Stokes system in $\mathbb{R}^3 \times \mathbb{R}_-$. Suppose also that \mathbf{v} satisfies (59) with some q and ℓ such that $\alpha \equiv \frac{3}{q} + \frac{2}{\ell} - 1 \in [0, 1]$. Let u be a Lipschitz bounded solution of (67)–(68) with $\varepsilon = +1$ in $\mathbb{R}^n \times \mathbb{R}_-$. If $u|_\Gamma = \text{const}$, then $u \equiv \text{const}$.*

Proof. Given R , we set $k = \frac{1}{2} \text{osc}_{Q_{2R}} u$ and consider two functions $V_1 = u - \inf_{Q_{2R}} u$ and $V_2 = \sup_{Q_{2R}} u - u$.

At least one of them satisfies (70) with $\mathfrak{N} = 2$. Therefore, Lemma 4.2 gives for this function the estimate (71), which implies (57) with $\varkappa_1 = 1 - \frac{1}{2} \widehat{\beta}_3(q, \ell, 2, \widehat{\mathcal{N}})$. Iteration of this inequality with respect to a suitable sequence $R_m \rightarrow \infty$ completes the proof. \square

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